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1992 J. Phys. A: Math. Gen. 25 1309

(http://iopscience.iop.org/0305-4470/25/5/032)

J. Phys. A: Math. Gen. 25 (1992) 1309–1327. Printed in the UK

Solvable systems of wave equations and non-Abelian Toda lattices

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Received 1 January 1991, in final form 24 July 1991

Abstract. This paper relates equivalence classes of coupled systems of N linear wave equations to motions of an $N \times N$ matrix dynamical system, the two-dimensional non-Abelian Toda lattice. In particular, the correspondence is shown to relate those coupled systems of wave equations with progressing-wave general solutions to motions of the finite non-Abelian Toda lattice with free ends, generalizing a known result for the $N = 1$ case. Some non-trivial motions of such Toda lattices are found, and the corresponding coupled wave equations and their progressing wave general solutions are given. Other consequences of the correspondence and possible applications of the progressing waves are discussed.

1. Introduction

In this paper an established relationship between second-order linear wave equations and two-dimensional Toda-lattice motions will be generalized to the case of *systems* of coupled linear wave equations and *non-Abelian* Toda-lattice motions. An application of this generalized relationship that we discuss in detail involves systems of coupled wave equations that are exactly solvable in terms of *progressing waves*, and we begin by reviewing this notion of exact solvability. There is no universally accepted definition of 'exactly solvable' for (systems of) differential equations. With reference to the Schrödinger equation, the notion is sometimes linked to a particular choice of the space of functions in which the solution is required to lie [1], while for Hamiltonian systems the fundamental concept of complete integrability provides a mathematically natural, but by no means unique, standard of solvability [2]. A precise and physically natural type of exact solvability for linear wave equations in two-dimensional spacetimes can be based on the concept of a progressing wave [3–5]. We specialize the definition in [5] and define a *progressing wave of finite order N* on a spacetime of two dimensions to be a member of a family of functions of the form

$$\phi = \sum_{n=0}^{N} U_n(x,t) f_n[S(x,t)] \tag{1.1}$$

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where the U_n are fixed functions on the spacetime, holding $S(x,t)$ constant defines a characteristic of the spacetime, and $f_n(z) = \mathrm{d}f_{n-1}(z)/\mathrm{d}z$, with $f_0(z)$ being any sufficiently differentiable function of one variable. We then define the general homogeneous second-order linear wave equation on a two-dimensional spacetime,

$$[g^{ab}(x^c)\nabla_a\nabla_b + 2A^a(x^c)\nabla_a + 2M(x^c)]\,\phi = 0 \tag{1.2}$$

where $a, b, c = 1, 2$, and g_{ab} is a Lorentzian metric with ∇_a the corresponding covariant derivative, to be *exactly solvable* when its general solution can be expressed as the sum of (two) such progressing waves. A simple example of such exact solvability is the equation $\partial^2_{uv}\phi = 0$ whose general solution is $\phi = a(v) + b(u)$, the sum of two 0th-order progressing waves. It is implicit in [3] and explicit in [4, 5] that this example can be generalized to the familiar family of equations

$$\partial^2_{vu}\phi + \frac{l(l+1)}{(v-u)^2}\,\phi = 0 \tag{1.3}$$

where l a non-negative integer, which has for a general solution

$$\phi = \sum_{n=0}^{l} \frac{c_{ln}}{(v-u)^n}\,\frac{\mathrm{d}^{l-n}}{\mathrm{d}v^{l-n}}a(v) + \sum_{n=0}^{l} \frac{c_{ln}}{(v-u)^n}\,\frac{\mathrm{d}^{l-n}}{\mathrm{d}u^{l-n}}b(u) \tag{1.4}$$

where $c_{ln} = (-1)^n(l+n)!/(l-n)!\,n!$, which is the sum of two progressing waves of order l.

It appears from [4] that in 1961 (1.3) and (1.4), and transformations of them, were the only known cases of such exact solvability for (1.2). The first attempt to substantially expand the family seems to have been made by Kundt and Newman in 1968 [6], who described such exactly solvable equations and their solutions as having the *characteristic propagation property*, since the waves propagate without the scattering off of characteristics that occurs in the general case. A few ostensibly new examples of exactly solvable wave equations given in [6] were later shown to be transformations of (1.3) and (1.4) [7], but a lasting contribution of [6] was to provide an effective test, applicable in principle to any example of (1.2), whose satisfaction guaranteed, and probably followed from, the exact solvability of the wave equation. To apply the test the example of (1.2) is put into a normal form with two functional coefficients and from these coefficients one generates in a prescribed way a doubly infinite *substitution sequence* of functions; if the sequence *terminates*, by producing a vanishing function in a finite number of steps, in both directions, then the original equation is certainly exactly solvable. In fact formulae for the progressing waves are obtained in terms of the functions of the doubly terminating sequence.

Following [6], the subject developed along two rather independent lines. One sequence of papers focused on finding essentially new but relatively small families of exactly solvable equations with useful applications in various areas of applied mathematics and physics. Chang and Janis [8, 9] had found families which enabled them to identify cosmological spacetimes whose most general purely gravitational perturbations were expressible in the nice closed form of progressing waves. This work was generalized by Couch and Torrence [7] to find many spherically symmetrical spacetimes on which the scalar wave equation was exactly solvable [10], and to greatly extend the results of Chang and Janis on purely gravitational perturbations of cosmological spacetimes [11]. In a different context, some isolated examples of exactly solvable acoustic equations in two-dimensional spacetimes had been given by Seymour and Varley [12], who used them as a starting point in constructing approximate solutions to more general acoustic equations, and these examples were generalized to find probably all self-adjoint acoustic equations in two-dimensional spacetimes that are exactly solvable in the progressing wave sense [13]. A paper by Gottlieb [14] on 'wake-free' solutions of the acoustic equation in two and three space dimensions was also clarified and generalized [15].

A second family of results flowed directly from those of Kundt and Newman [6] and results in this paper extend that work. It was clear from the start that to find an example of (1.2) that yielded a doubly terminating substitution sequence of total length k amounted to finding a particular solution of an ostensibly formidable $(2k)$th-order non-linear partial differential equation in two independent variables. The papers reviewed above were based, in effect, on guessing small families of solutions to that equation. Then in [16, 17] it was shown that the nonlinear system in question was precisely the finite two-dimensional free-ended Toda lattice [18]. Since the general solution to that system was known [19], one immediatly had the general solution to the double termination condition of Kundt and Newman, and that probably all examples of (1.2) that are exactly solvable in the progressing wave sense had been found. It also turned out that some specific families of wave equations already known to be exactly solvable, of which (1.3) is the simplest example, could be used to isolate some particular families of solutions of the finite free-ended Toda lattice that may not have been recognized earlier [20].

It is the main purpose of this paper to generalize to systems of coupled linear wave equations the basic results reviewed above for single wave equations. For this purpose we will need the matrix generalization of the standard Toda lattice, usually referred to as the non-Abelian Toda lattice. Toda [18] initiated the study of the system of equations

$$\frac{\mathrm{d}^2 y_k}{\mathrm{d}t^2} = \mathrm{e}^{-(y_k - y_{k-1})} - \mathrm{e}^{-(y_{k+1}-y_k)} \qquad k \in \mathrm{Z} \tag{1.5}$$

usually referred to as the one-dimensional scalar (Abelian) Toda lattice. Useful alternative forms of (1.5) are obtained if we define $r_k = y_{k+1} - y_k$, in which case (1.5) implies that

$$\frac{\mathrm{d}^2 r_k}{\mathrm{d}t^2} = -\mathrm{e}^{-r_{k-1}} + 2\mathrm{e}^{-r_k} - \mathrm{e}^{-r_{k+1}} \qquad k \in \mathrm{Z} \tag{1.6}$$

or introduce

$$n_k = -\frac{\mathrm{d}y_k}{\mathrm{d}t} \qquad m_k = \mathrm{e}^{-(y_k - y_{k-1})} \qquad k \in \mathrm{Z} \tag{1.7}$$

in which case (1.5) implies that

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} n_k &= m_{k+1} - m_k \\ \frac{\mathrm{d}}{\mathrm{d}t} m_k &= m_k (n_k - n_{k-1}) \end{aligned}\right\} \qquad k \in \mathrm{Z}\,. \tag{1.8}$$

Toda interpreted (1.5) mechanically, as a one dimensional array of molecules experiencing nearest-neighbour exponential interactions. This is satisfying for the infinite lattice (1.5) and for its periodic version, which is that specialization of (1.5) characterized by assuming that $y_k = y_{n+k}$ for all $k \in \mathrm{Z}$, where n is some positive integer. The finite Toda lattice with free ends, which has a particular role in this paper, is defined by considering in (1.5) only those equations with $k_0 \leqslant k \leqslant k_1$, for some integers k_0 and k_1, with $y_{k_0-1} \to -\infty$ and $y_{k_1+1} \to +\infty$, and in this case a mechanical interpretation is less satisfactory, since the molecules with minimal and maximal indices experience negative and positive accelerations, respectively, regardless of their displacements relative to their (single) nearest neighbours. In what follows, we shall want to allow m_k of both signs, which is not easily accomodated by (1.5)–(1.7), and, although we may refer to the displacement interpretation for motivational purposes, we find it convenient to work with the form (1.8).

The generalization of (1.8) given [21] by

$$\begin{aligned} (\partial_t + \partial_x)\, n_k &= \partial_v n_k = m_{k+1} - m_k \\ (\partial_t - \partial_x)\, m_k &= -\partial_u m_k = m_k\,(n_k - n_{k-1}) \end{aligned} \qquad k \in \mathrm{Z} \qquad (1.9)$$

where $v+u = x$, $v-u = t$, is usually referred to as the two-dimensional Abelian (scalar) Toda lattice, and has been extensively studied [19, 22]. A mechanical interpretation of (1.9) as a plane array of exponentially interacting stretched strings satisfying

$$\partial^2_{uv} y_k = \mathrm{e}^{-(y_{k+1}-y_k)} - \mathrm{e}^{-(y_k-y_{k-1})} \qquad k \in \mathrm{Z} \qquad (1.10)$$

with

$$n_k = \partial_u y_k \qquad m_k = \mathrm{e}^{-(y_k-y_{k-1})} \qquad k \in \mathrm{Z} \qquad (1.11)$$

is possible [17], and the system arises in other physical contexts [23]. An alternative generalization of (1.8) is to the one-dimensional non-Abelian (matrix) Toda lattice

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} N_k &= M_{k+1} - M_k \\ \frac{\mathrm{d}}{\mathrm{d}t} M_k &= N_k M_k - M_k N_{k-1} \end{aligned} \qquad k \in \mathrm{Z} \qquad (1.12)$$

where the dynamical variables are $N \times N$ matrices. This is a non-trivial generalization of (1.8), since the matrices are in general non-commutative, and it is known to arise naturally in a few mathematical and physical settings [24, 25]. A useful survey of antecedents and possible applications of these systems has been given by Popowicz [26], who also discusses an obvious further generalization of (1.8) that combines (1.9) and (1.12), the two-dimensional non-Abelian (matrix) Toda lattice

$$\begin{aligned} (\partial_t + \partial_x)\, N_k &= \partial_v N_k = M_{k+1} - M_k \\ (\partial_t - \partial_x)\, M_k &= -\partial_u M_k = N_k M_k - M_k N_{k-1} \end{aligned} \qquad k \in \mathrm{Z} \qquad (1.13)$$

a system that has been studied by Mikhailov [27] and Andreev [28, 29], and has some physical significance [30, 31]. The periodic and finite specializations of (1.9) and (1.12) are included in those of (1.13), namely $N_k = N_{k+n}$, $M_k = M_{k+n}$ for all $k \in \mathrm{Z}$, n being a positive integer, and $k_0 \leqslant k \leqslant k_1$, with $M_{k_0} = M_{k_1+1} = 0$, so that $\partial_v N_{k_0} = M_{k_0+1}$ and $\partial_v N_{k_1} = -M_{k_1}$, respectively.

In the next section we review the correspondence between the Abelian lattice motions and single linear wave equations. We also note an aspect of the concept of formal self-adjointness for such equations arising under this correspondence, that leads in a natural way to an alternative kind of formal self-adjointness for single linear wave equations, which we shall call *almost self-adjointness*, and which may not have been recognized before. In section 3 we derive the equations relating the motions of non-Abelian Toda lattices to systems of linear wave equations, which are our basic result. It should be emphasized that these results relate to arbitrary non-Abelian Toda lattices and general coupled systems of wave equations, although it is the finite free-ended lattices that are relevant to the considerations of exactly solvable linear wave equations emphasized in this paper. In section 4 the generalization of this latter relationship from the Abelian to the non-Abelian setting is given. In addition some new particular motions of two- and three-element free-ended non-Abelian Toda lattices for the case of symmetric 2×2 matrices are given, along with the corresponding pairs of coupled wave equations and their progressing-wave solutions. We are not aware of any prior non-trivial examples of this in the literature.

In the concluding section some open problems are discussed. The specific examples of exactly solvable pairs of coupled equations given in section 4 were designed to be self-adjoint, or almost self-adjoint systems, and corresponded to non-Abelian generalizations of two-, or three-element anti-symmetrical Abelian lattice motions, with or without centre, respectively. Somewhat surprisingly, when one attempts to extend this relationship to the case of longer Toda chains, one meets with novelties peculiar to the non-Abelian case whose full significance is as yet unclear. These matters are discussed in some detail. In addition we outline the possible application of the progressing wave aspects of this work to the study of gravitational perturbations of particular curved spacetime backgrounds. Finally, in the Abelian case, some exactly solvable wave equations reduce to Schrödinger equations with reflectionless potentials, and we discuss the possibility that an analogous reduction in the non-Abelian case might give new and interesting sets of reflectionless potentials for multichannel scattering processes.

2. The scalar case

In a 1968 paper, Kundt and Newman [6] initiated a search for homogeneous second-order linear wave equations in two-dimensional spacetimes whose solutions propagate without the continuous backward scattering that is generic for such equations. To this end it was noted in [6] that each such equation can be put into either one of the two normal forms $(\partial_v a\, \partial_u - b)\,\phi = 0$ and $(\partial_u c\, \partial_v - d)\,\psi = 0$, where a, b, c, and d depend in general on u and v. If we define $j_0 = a$, $j_1 = b$, $\phi_0 = \phi$, and inductively define $\{j_k\}_{k\in Z}$, $\{\phi_k\}_{k\in Z}$ by

$$\begin{aligned} j_{k+1}/j_k &= j_k/j_{k-1} - \partial_u\left[(\partial_v j_k)/j_k\right] \\ j_{k+1}\phi_{k+1} &= j_k\, \partial_u \phi_k \end{aligned} \qquad k \in \mathrm{Z} \tag{2.1}$$

assuming of course $j_k \neq 0$ for all k, we obtain a countable set of equations

$$(\partial_v j_k \partial_u - j_{k+1})\,\phi_k = 0 \qquad k \in \mathrm{Z}\,. \tag{2.2}$$

Similarly, if we define $l_0 = c$, $l_{-1} = d$, $\psi_0 = \psi$, and inductively define $\{l_k\}_{k\in Z}$ $\{\psi_k\}_{k\in Z}$ by

$$\begin{aligned} l_{k-1}/l_k &= l_k/l_{k+1} - \partial_v\left[(\partial_u l_k)/l_k\right] \\ l_{k-1}\psi_{k-1} &= l_k\,\partial_v\psi_k \end{aligned} \qquad k \in Z \tag{2.3}$$

we obtain a second countable set

$$(\partial_u l_k \partial_v - l_{k-1})\,\psi_k = 0 \qquad k \in Z\,. \tag{2.4}$$

We shall refer to the equations (2.2) as being in v-normal form, and those given by (2.4) as being in u-normal form, and it is not hard to confirm that for all $k \in Z$ the kth equation in (2.2), corresponding to the coefficients j_k, j_{k+1}, and the kth equation in (2.4), corresponding to the coefficients l_k, l_{k-1}, are the same equation in v-normal form and u-normal form, respectively, where

$$j_k l_k = 1 \qquad \phi_k = l_k\psi_k \qquad k \in Z\,. \tag{2.5}$$

It will be shown in section 3 that in a more general setting the equations within the set (2.2) (respectively, (2.4)) are equivalent in the sense that a solution of any one of them generates a solution of every one of them through (2.1), (2.2) and (2.5) (respectively, (2.3), (2.4) and (2.5)), and we shall designate the corresponding sets of equivalence classes of v-normal form equations $\mathcal{V} \equiv \{\mathcal{V}_{(\alpha)}\}_{\alpha\in A}$, and of the u-normal form equations $\mathcal{U} \equiv \{\mathcal{U}_{(\alpha)}\}_{\alpha\in A}$, where A is an appropriate set of indices. It should be emphasized that these results, taken from [6] and [17], hold with k ranging over both negative and positive integers.

The connection between the set of linear wave equations in two-dimensional space-times and the set of Abelian Toda lattice motions can be based on either of the sets of equivalence classes of normal form wave equations $\mathcal{U}$ and $\mathcal{V}$. A particular element $\mathcal{U}_{(\alpha)}$ of $\mathcal{U}$ corresponds to an equivalence class $\{\mathcal{U}_{(\alpha)k}\}_{k\in Z}$ of u-normal form wave equations, and thus to the corresponding sequence $\{l_{(\alpha)k}\}_{k\in Z}$ of coefficient functions, with $\mathcal{U}_{(\alpha)k} \sim (l_{(\alpha)k}, l_{(\alpha)k-1})$. If we now define

$$m_{k+1} = l_k/l_{k+1} \qquad n_k = (\partial_u l_k)/l_k \qquad k \in Z \tag{2.6}$$

it follows from (2.3) that the first of (1.9) is satisfied, while the second holds identically, so from $\mathcal{U}_{(\alpha)}$ we have generated a solution of the two-dimensional Abelian Toda lattice, defining a map $T_{\mathcal{M}} : \mathcal{U} \to \mathcal{M}$ into the set $\mathcal{M}$ of motions in the (m_k, n_k) representation. Conversely, if we begin with an element of $\mathcal{M}$, (2.6) determines a sequence $\{l_k\}_{k\in Z}$, up to an arbitrary single multiplicative function of the coordinate v, and thus an element of $\mathcal{U}$, as we can see from (2.4) that such a freedom in $\{l_k\}$ does not affect the wave equations. Thus, the map $T_{\mathcal{M}}$ is in fact a *bijection*. This map is structure preserving, *in the sense that the equivalence relation defining* $\mathcal{U}_{(\alpha)}$ is mapped to the dynamics defining the motion of the lattice. Clearly, there exists another representation of the same correspondence appropriate to the classes of v-normal form equations $\{\mathcal{V}_{(\alpha)}\}_{\alpha\in A}$. If we define

$$q_k = j_{k+1}/j_k \qquad p_k = (\partial_v j_k)/j_k \qquad k \in Z \tag{2.7}$$

then we have

$$\begin{aligned} (\partial_t - \partial_x)\, p_k &= -\partial_u p_k = q_k - q_{k-1} \\ (\partial_t + \partial_x)\, q_k &= \partial_v q_k = q_k\, (p_{k+1} - p_k) \end{aligned} \qquad k \in \mathrm{Z} \tag{2.8}$$

the first being (2.1), and the second a consequence of (2.7). Thus (2.8) is a description of the Abelian Toda lattice, alternative to (1.9), and (2.7) defines a bijection $T_{\mathcal{P}} : \mathcal{V} \to \mathcal{P}$ between its set of motions $\mathcal{P}$ in the (q_k, p_k) representation and $\mathcal{V}$.

A provocative observation follows from the existence of a subset of the Toda lattice motions which might naturally be called anti-symmetrical, and that can be divided into the classes

$$y_{-k} = -y_k \qquad k \in \mathrm{Z} \tag{2.9}$$

(with, in particular, $y_0 = 0$), and

$$y_{-k} = -y_{k-1} \qquad k \in \mathrm{Z} \tag{2.10}$$

centred at the zeroth element, and between the zeroth element and the element indexed by -1, respectively. It is easy to see from (1.7) that (2.9) and (2.10) correspond to

$$n_{-k} = -n_k \qquad m_{-k} = m_{k+1} \qquad k \in \mathrm{Z} \tag{2.11}$$

and

$$n_{-k} = -n_{k-1} \qquad m_{-k} = m_k \qquad k \in \mathrm{Z} \tag{2.12}$$

respectively, and thus to

$$l_{-k} = 1/l_k \qquad k \in \mathrm{Z} \tag{2.13}$$

(in particular, $l_0 = 1$), and

$$l_{-k} = 1/l_{k-1} \qquad k \in \mathrm{Z} \tag{2.14}$$

respectively, up to the freedom available in the u-normal forms of multiplying all l_k by a function of v, as discussed after (2.6). Thus, we obtain equivalence classes of equations $\mathcal{U}_{(\alpha)}$ in $\mathcal{U}$ each including as a representative equation

$$(\partial_u \partial_v - l_{-1})\, \psi_0 = 0 \tag{2.15}$$

and

$$(\partial_u l_0 \partial_v - 1/l_0)\, \psi_0 = 0 \tag{2.16}$$

respectively. Now (2.15) with arbitrary l_{-1} is the most general self-adjoint wave equation, since any constant l_0 can be transformed to 1, in the formal sense in which we do not consider the issue of boundary conditions on ψ_0, so we see that ***anti-symmetrical Abelian Toda-lattice motions with a centre element*** (2.9) correspond to self-adjoint wave equations (2.15). But, similarly, ***anti-symmetrical Abelian Toda lattice motions*** *without a centre element* (2.10) correspond to equations of the type of (2.16), which we will refer to as *almost self-adjoint wave equations*. Notice that these correspondences hold in the opposite direction as well since, given wave equations satisfying (2.15) or (2.16), the recursion relation in (2.3) generates l_ks all of which satisfy (2.13) or (2.14), respectively; in other words, the antisymmetry conditions are fully compatible with the Toda lattice dynamics. Thus the map $T_{\mathcal{M}}^{-1} : \mathcal{M} \to \mathcal{U}$ has called attention to a family of wave equations (2.16) that might be expected to share many properties with the self-adjoint equations (2.15), but may not have received serious study before.

To date the most useful set of results obtained from $T_{\mathcal{P}}^{-1} : \mathcal{P} \to \mathcal{V}$ concerns wave equations with progressing-wave solutions [16, 17]. Given a wave equation in v-normal form,

$$(\partial_v j_0 \partial_u - j_1)\,\phi_0 = 0 \tag{2.17}$$

we say [6] that its substitution sequence $\{j_k\}_{k\in\mathbb{Z}}$ is *double terminating* when $j_{k_1+1} = 0$ and $l_{k_0-1} = 0$ for some $k_1 \geqslant 0$, $k_0 \leqslant 0$. But it was shown in [6], using (2.1), (2.3) and (2.5), that in this case (2.17) is solved by $\phi_0 = \phi_A + l_0\psi_R$, where

$$\begin{aligned} \phi_A &= (j_1^{-1}\partial_v j_1)(j_2^{-1}\partial_v j_2)\cdots(j_{k_1}^{-1}\partial_v j_{k_1})a(v) \\ \psi_R &= (l_{-1}^{-1}\partial_u l_{-1})(l_{-2}^{-1}\partial_u l_{-2})\cdots(l_{k_0}^{-1}\partial_u l_{k_0})b(u) \end{aligned} \tag{2.18}$$

with $a(v)$, $b(u)$ arbitrary functions of one variable, where there are k_1 and $-k_0$ partial differentiations, respectively, and they operate on everything to their right. Thus (2.17) has a general solution expressible as sums of examples of the mathematically and physically simple progressing waves of finite order defined in the first section. Now the double termination condition on $\{j_k\}_{k\in\mathbb{Z}}$ corresponds, under $T_{\mathcal{P}} : \mathcal{V} \to \mathcal{P}$, to a motion of a finite Toda lattice with free ends. Thus each such Toda lattice motion picks out an equivalence class of $k_1 - k_0 + 1$ linear wave equations, with individual equations corresponding to adjacent pairs (j_k, j_{k+1}) of coefficient functions in the substitution sequence, each of which has a progressing wave general solution. In fact, a general solution of the free-ended finite Toda lattice is known [19], so $T_{\mathcal{P}}^{-1} : \mathcal{P} \to \mathcal{V}$ yields an explicit construction of all linear wave equations in two-dimensional spacetimes with progressing wave general solutions, to the extent that the known general solution of the (non-linear) Toda lattice equations includes all solutions, and that no non-terminating sequence can give rise to linear wave equations with progressing wave general solutions. This probably gives a full solution to the problem that motivated the paper [6] which was the starting point of this section, although the results being reviewed here go somewhat beyond the original work of Kundt and Newman. Naturally these progressing wave results could equally well have been derived with reference to u-normal form equations and $T_{\mathcal{M}}^{-1} : \mathcal{M} \to \mathcal{U}$.

3. The non-Abelian case

We begin with the matrix generalization of (1.2), i.e. with the system of linear wave equations

$$(g^{ab}\nabla_a\nabla_b + 2A^a\nabla_a + 2M)\,\Phi = 0 \tag{3.1}$$

where $a,b = 1,2$, g_{ab} is a Lorentzian metric with ∇_a the corresponding covariant derivative, A^a and M are $N \times N$ matrices, and Φ is an $N \times 1$ matrix. Although it is easy to see that in the case $N = 1$, (3.1) is a particular geometrical representation of the most general second-order linear wave equation in a two-dimensional spacetime, it is obvious that for $N \geqslant 2$, (3.1) is a special family of systems of such equations, which we choose to cast in a geometrical form. Systems of equations of this kind do arise in physical problems. Consider for example a vector-valued complex field Φ (Higgs field) coupled to a non-Abelian gauge field, described by the (anti-Hermitian) matrix-valued gauge potential A_a. The Lagrangian for such a field is of the form $\mathcal{L} = (D_a\Phi)^{\dagger}(D^a\Phi) + \Phi^{\dagger}\mu\,\Phi$, where $D_a\Phi = (\nabla_a + A_a)\,\Phi$ is the gauge-covariant derivative, and μ a mass matrix. Then the field equation for Φ which we obtain by varying the action is $D^aD_a\Phi - \mu\,\Phi = 0$, which is of the form (3.1) if we set $2M = \nabla_a A^a + A_a A^a - \mu$.

It is not necessary, but convenient, to describe our calculations in language appropriate to gauge theories. If we gauge the field Φ by the transformation

$$\Phi' = G(x^a)\,\Phi \tag{3.2}$$

it follows that (3.1) is replaced by

$$(g^{ab}\nabla_a\nabla_b + 2A'^a\nabla_a + 2M')\,\Phi' = 0 \tag{3.3}$$

where $A'^a = G\,A^aG^{-1} + g^{ab}G\,\partial_b G^{-1}$, and $M' = G\,M\,G^{-1} + \frac{1}{2}\,g^{ab}G\,\nabla_a\nabla_b G^{-1}$. In two dimensions it is always possible to choose coordinates in which g_{ab} is manifestly conformally related to the Minkowski metric η_{ab}, and we do so, introducing null coordinates u and v in which

$$g_{ab} = g^{1/2}\eta_{ab} = g^{1/2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.4}$$

so that (3.1) becomes

$$(\partial^2_{uv} + A_u\partial_u + A_v\partial_v + g^{1/2}M)\,\Phi = 0 \tag{3.5}$$

where

$$A_u = g^{1/2}A^v \qquad A_v = g^{1/2}A^u\,.$$

We can work, interchangeably, in the two null gauges by either choosing G so that

$$A'_u = G^{-1}A_uG - G^{-1}\partial_uG = 0 \tag{3.6}$$

in which case (3.5) takes the *v-gauge* form

$$(\partial_v J_0 \partial_u - J_1)\,\Phi_0 = 0 \tag{3.7}$$

with J_0 defined by $\partial_v J_0 = J_0 A'_v$, $J_1 = -J_0\,g^{1/2}M'$, and $\Phi_0 = \Phi'$, or by choosing G so that

$$A'_v = G^{-1}A_vG - G^{-1}\partial_vG = 0 \tag{3.8}$$

in which case (3.5) takes the *u-gauge* form

$$(\partial_u L_0 \partial_v - L_{-1})\Psi_0 = 0 \tag{3.9}$$

with L_0 defined by $\partial_u L_0 = L_0 A'_u$, $L_{-1} = -L_0 g^{1/2} M'$, and $\Psi_0 = \Psi'$, where the $-1, 0, +1$ subscripts have been introduced in anticipation of subsequent developments. Equations (3.7) and (3.9) are the systemic generalizations of the $k = 0$ cases of (2.2) and (2.4), the v-normal form and u-normal form equations of section 2, respectively, and we will now generalize the calculations outlined there to obtain the analogues of (2.2) and (2.4) for all $k \in \mathrm{Z}$.

We consider first the v-gauge equation (3.7) and inductively define $\{J_k\}_{k\in \mathrm{Z}}$ by

$$\begin{aligned} J_k^{-1} J_{k+1} &= J_{k-1}^{-1} J_k - \partial_u (J_k^{-1} \partial_v J_k) \\ J_{k+1}\Phi_{k+1} &= J_k\, \partial_u \Phi_k \qquad k \in \mathrm{Z} \end{aligned} \tag{3.10}$$

generalizing (2.1). Solving (3.7) for Φ_0, applying ∂_u to both sides of the resulting equation, and using $\partial_u\partial_v = \partial_v\partial_u$, the second equation in (3.10) results in $[\partial_v J_1 \partial_u - J_2]\Phi_1 = 0$, given the first equation of (3.10). Iterating in both directions with respect to k, we generate the set of equations

$$(\partial_v J_k \partial_u - J_{k+1})\Phi_k = 0 \qquad k \in \mathrm{Z}\,. \tag{3.11}$$

Beginning with the u-gauge equation (3.9) and defining $\{L_k\}_{k\in \mathrm{Z}}$ and $\{\Psi_k\}_{k\in \mathrm{Z}}$ by

$$\begin{aligned} L_k^{-1} L_{k-1} &= L_{k+1}^{-1} L_k - \partial_v (L_k^{-1} \partial_u L_k) \\ L_{k-1}\Psi_{k-1} &= L_k\, \partial_v \Psi_k \end{aligned} \qquad k \in \mathrm{Z} \tag{3.12}$$

we similarly generate the set of u-gauge equations

$$(\partial_u L_k \partial_v - L_{k-1})\Psi_k = 0 \qquad k \in \mathrm{Z}\,. \tag{3.13}$$

As before, we denote the sets of equivalence classes of systems of wave equations defined by (3.11) and (3.12), $\mathcal{V} = \{\mathcal{V}_{(\alpha)}\}_{\alpha\in A}$ and $\mathcal{U} = \{\mathcal{U}_{(\alpha)}\}_{\alpha\in A}$, respectively. If (3.7) and (3.9) represent the same set of field equations in the different gauges, there should be a simple connection between them, and similarly for (3.11) and (3.13), pairwise. In fact one can show directly that the kth elements of (3.11) and (3.13), characterized by the pairs of coefficient functions (J_k, J_{k+1}) and (L_k, L_{k-1}), respectively, are the same equation in v-gauge and u-gauge, respectively, given

$$J_k L_k = \mathrm{I} \qquad \Phi_k = J_k^{-1}\Psi_k \qquad k \in \mathrm{Z} \tag{3.14}$$

which generalizes (2.5). Applying the general formalism of gauge theories we can pass from the coefficients in (3.11) and (3.13) to the corresponding gauge potentials via $\partial_v J_k = J_k A'_v$, $J_{k+1} = -J_k g^{1/2} M'$, $\partial_u L_k = L_k A'_u$, $L_{k-1} = -L_k g^{1/2} M'$, and thence to the gauge fields $F_{ab} = \nabla_a A_b - \nabla_b A_a + [A_a, A_b]$ to obtain

$$\begin{aligned} F^k_{uv} &= -F^k_{vu} = \partial_u (J_k^{-1} \partial_v J_k) \\ G^k_{vu} &= -G^k_{uv} = \partial_v (L_k^{-1} \partial_u L_k) \end{aligned} \qquad k \in \mathrm{Z} \tag{3.15}$$

for the non-vanishing components of the gauge field in the v-gauge and the u-gauge, respectively, and it follows from general considerations, or can be shown directly, that

$$F^k_{uv} = J_k^{-1} G^k_{uv} J_k \qquad k \in \mathrm{Z}\,. \tag{3.16}$$

It is satisfying that these gauge field components, (3.15), occur naturally in our basic generating formulae (3.10), (3.12). Actually, the normal forms (3.11) and (3.13) do not completely fix the gauge. It is easy to see that the transformation to

$$J'_k = U(u)\, J_k V(v) \qquad \Phi'_k = V^{-1}(v)\, \Phi_k \qquad k \in \mathrm{Z} \tag{3.17}$$

leaves (3.10) and (3.11) unchanged in form, while the transformation to

$$L'_k = V(v)\, L_k U(u) \qquad \Psi'_k = U^{-1}(u)\, \Psi_k \qquad k \in \mathrm{Z} \tag{3.18}$$

leaves (3.12) and (3.13) similarly unchanged, where U and V are arbitrary matrix-valued functions of one variable, so that (3.17) and (3.18) are the residual gauge freedoms in the v-gauge and u-gauge equations (3.6) and (3.8), respectively.

If we now define

$$M_{k+1} = L_{k+1}^{-1} L_k \qquad N_k = L_k^{-1}\, \partial_u L_k \qquad k \in \mathrm{Z} \tag{3.19}$$

it follows that the two-dimensional non-Abelian Toda-lattice equations (1.13) are satisfied, the first as a consequence of the first equation of (3.12) and the second identically. If we let $\mathcal{M}$ represent the non-Abelian Toda-lattice motions, as in the Abelian case, (3.19) defines a map $T_{\mathcal{M}} : \mathcal{U} \to \mathcal{M}$ generalizing that given in section 2 for the Abelian case. Once again it is, up to a trivial gauge transformation included in (3.17), a bijection. Working instead in the v-gauge and defining

$$Q_k = J_k^{-1} J_{k+1} \qquad P_k = J_k^{-1}\, \partial_v J_k \qquad k \in \mathrm{Z} \tag{3.20}$$

we find similarly a matrix generalization of (2.8),

$$\begin{aligned} (\partial_t - \partial_x)\, P_k &= -\partial_u P_k = Q_k - Q_{k-1} \\ (\partial_t + \partial_x)\, Q_k &= \partial_v Q_k = Q_k P_{k+1} - P_k Q_k \end{aligned} \qquad k \in \mathrm{Z} \tag{3.21}$$

which is an alternative representation of the two-dimensional non-Abelian Toda lattice, and we have a bijection up to gauge $T_{\mathcal{P}} : \mathcal{V} \to \mathcal{P}$, where $\mathcal{P}$ is the set of motions of the non-Abelian Toda lattice in the representation (3.21).

It is worth noting that one could write down a variety of distinct non-Abelian generalizations of (1.9), and it is satisfying that the above derivation led to (1.13), the generalization that has already appeared in the literature in several contexts [26–31].

4. Systems with progressing-wave solutions

The correspondence between motions of finite free-ended Abelian Toda lattices and single wave equations with progressing-wave general solutions carries over nicely to the case of systems. We define a sequence $\{J_k\}_{k\in\mathrm{Z}}$ to be *doubly terminating* when $J_{k_1+1} = 0$, $L_{k_0-1} = 0$ for some integers $k_1 \geqslant 0$, $k_0 \leqslant 0$. It follows immediately from

(3.10), (3.12) and (3.14) that in this case the equation $(\partial_v J_0 \partial_u - J_1)\Phi_0 = 0$ is solved by $\Phi_0 = \Phi_A + L_0 \Psi_R$, where

$$\begin{aligned} \Phi_A &= (J_1^{-1}\partial_v J_1)(J_2^{-1}\partial_v J_2)\cdots(J_{k_1}^{-1}\partial_v J_{k_1})A(v) \\ \Psi_R &= (L_{-1}^{-1}\partial_u L_{-1})(L_{-2}^{-1}\partial_u L_{-2})\cdots(L_{k_0}^{-1}\partial_u L_{k_0})B(u) \end{aligned} \tag{4.1}$$

where $A(v)$ and $B(u)$ are $N \times 1$ matrices of arbitrary functions of one variable, there are k_1 and $-k_0$ partial differentiations, respectively, and they operate on everything to their right. Clearly, every system of equations in the same equivalence class can be similarly satisfied. But (4.1) is a systemic generalization of progressing waves, and we have found a progressing-wave general solution for these systems of equations. As before, double termination for the sequence $\{J_k\}_{k\in Z}$ corresponds under $T_{\mathcal{P}} : \mathcal{V} \to \mathcal{P}$ to a motion of a finite non-Abelian Toda lattice with free ends.

The construction of explicit examples of such motions is at present a non-trivial exercise compared to the Abelian case, where by building on earlier work [19] an explicit construction of probably all linear wave equations with progressing-wave solutions were constructed [17]. In the non-Abelian case the complete integrability of the finite Toda lattice has been established in both the periodic [32] and non-periodic [33] cases, but this kind of integrability does not in itself provide an explicit construction of solutions of the dynamical system. Bäcklund transformations for a restricted class of non-Abelian free-ended lattices, and some particular motions for that same class, have been given by Andreev [28, 29], but from the point of view of the corresponding coupled wave equations Andreev's restricted class is not a particularly natural one. We will derive other special solutions here that correspond to (exactly solvable) truly coupled systems of linear wave equations that are particularly basic in the sense that they are formally self-adjoint.

We work with the representation (3.20), (3.21) of the lattice dynamics and, for our first example, we start with the assumption that the motion satisfies

$$J_{-k} = J_{k+1}^{-1} \qquad k \in Z \tag{4.2}$$

which is the natural generalization to the matrix case of an anti-symmetrical motion centred between the zeroth and first elements. Thus it is the matrix generalization corresponding to (2.10), but in the v-gauge rather than the u-gauge, and differently centred. The consistency of this ansatz with the dynamical equations is not a foregone conclusion in the non-Abelian case. Solving (3.10) for J_2 and J_{-1}^{-1} yields

$$\begin{aligned} J_2 &= J_1\left[-\partial_u(J_1^{-1}\partial_v J_1) + J_1^2\right] \\ J_{-1}^{-1} &= \left[\partial_u(J_1\partial_v\ J_1^{-1}) + J_1^2\right] J_1 \end{aligned} \tag{4.3}$$

where (4.2) for $k = 0$ has been used, and it is easy to derive that (4.2) for $k = 1$ implies that

$$(\partial_v J_1)\ J_1^{-1}\ (\partial_u J_1) = (\partial_u J_1)\ J_1^{-1}\ (\partial_v J_1) \tag{4.4}$$

which is identically satisfied only in the Abelian case. Imposing in addition the condition that the lattice be just two elements long with free ends, i.e. $J_2 = J_{-1}^{-1} = 0$, leads to

$$\partial_u(J_1^{-1}\partial_v J_1) = J_1^2 \tag{4.5}$$

and in fact (4.4) and (4.5) are equivalent to (4.3) supplemented by the conditions for double termination. We shall set up a second example as well, since its mathematical treatment turns out to be simply related to that of the first one, at least in the case to which we will be specializing. We first assume that

$$J_{-k} = J_k^{-1} \qquad k \in \mathbb{Z} \tag{4.6}$$

the matrix generalization of an anti-symmetrical motion centred on the zeroth element. *It is easy to show that combining* (4.6) *for* $k = 0, 1$ with the dynamics again implies (4.4), and imposing $J_2 = J_{-2}^{-1} = 0$ specifies our second dynamical problem, which is equivalent to (4.4) and

$$\partial_u(J_1^{-1}\partial_v J_1) = J_1\,. \tag{4.7}$$

It is worth noting that in the Abelian case, where (4.4) is identically satisfied, (4.7) reduces to the familiar Liouville equation, $\partial^2_{uv} f = e^f$, with the general solution $f = 2U'(u)V'(v)/[U(u) + V(v)]^2$. Thus (4.7) is a natural matrix generalization of that equation.

We next assume in both examples that

$$J_1(u, v) = J_1(u + v) = J_1(x) \tag{4.8}$$

in which case (4.4) is identically satisfied, and our two examples reduce to

$$\partial_x(J_1^{-1}\partial_x J_1) = J_1^2 \tag{4.9}$$

and

$$\partial_x(J_1^{-1}\partial_x J_1) = J_1 \tag{4.10}$$

respectively. As a further simplification we assume in both cases, with an eye to the corresponding set of coupled wave equations, that J_1 is a symmetric matrix.

We shall solve the three-element example, which has been reduced to solving (4.10), and subsequently use a simple transformation to obtain solutions for the two-element equation (4.9). The first step is to linearize the problem. It follows directly from (4.10) and its transpose that

$$J_1^{-1}(\partial_x J_1) - (\partial_x J_1)J_1^{-1} = C \tag{4.11}$$

where C is a constant anti-symmetric matrix. We assume that $C \neq 0$, as it can be shown that this guarantees that our non-Abelian lattice motion is not equivalent, by diagonalizing J_1 by a *similarity transformation*, to merely juxtaposing uncoupled Abelian lattice motions. If we now define

$$F = J_1^{-1}(\partial_x J_1)J_1^{-1} = -\partial_x(J_1^{-1}) \tag{4.12}$$

it follows from (4.10) that F satisfies both

$$F^2 = 2J_1^{-1} + J_1^{-1}BJ_1^{-1} \tag{4.13}$$

where B is a constant symmetric matrix, and

$$\partial_x F + CF + I + BJ_1^{-1} = 0 \tag{4.14}$$

where I is the identity matrix. Then by differentiating (4.14) we find that F satisfies the second-order constant coefficient linear matrix differential equation

$$\partial_x^2 F + C\partial_x F - BF = 0\,. \tag{4.15}$$

In principle, to obtain a solution to (4.10) we now solve the linear equation (4.15) for F and substitute the result into (4.11), (4.13) and (4.14) to eliminate any spurious solutions, and to obtain J_1. Thus we have effectively linearized the solving of the non-linear equation (4.10), although we shall see that the last step results in non-linear algebraic constraints on the constant components b_{ij} of B and the constants of integration introduced when (4.15) is solved.

The algebraic manipulations are tractable if we make the final simplifying assumption that the matrices are 2×2, which means that the corresponding system of wave equations comprises just two coupled equations. Most of the solutions of (4.10) in this case can be obtained as follows. We first rescale both the coordinate x and J_1 so that (4.10) is preserved and at the same time we have

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.16}$$

Next we note that under a constant similarity transformation on J_1, C, and B, equations (4.10)–(4.14) are invariant, and that in the 2×2 case the similarity matrix can be chosen such that B is diagonal and (4.16) is unchanged. Thus we can take B to be diagonal without loss of generality. If we define

$$d = 1 + (b_{11} - b_{22})^2 - 2(b_{11} + b_{22}) \tag{4.17}$$

then the four roots of the characteristic polynomial of the first-order system equivalent to the first column of (4.15) are given by $\pm m$ and $\pm n$, where m, n are the (possibly complex) quantities given by

$$m = \left[\tfrac{1}{2}(b_{11} + b_{22} - 1 + \sqrt{d})\right]^{1/2} \qquad n = \left[\tfrac{1}{2}(b_{11} + b_{22} - 1 - \sqrt{d})\right]^{1/2}. \tag{4.18}$$

Precisely when the four roots are distinct, i.e. when $d \neq 0$ and $\det B \neq 0$, the elements f_{ij} satisfying (4.15) can be expressed in the form

$$f_{12} = a_1 \sinh mx + a_2 \cosh mx + a_3 \sinh nx + a_4 \cosh nx \tag{4.19}$$

and

$$\begin{aligned} f_{11} &= [-\partial_x^3 f_{12} - (1 - b_{22})\partial_x f_{12}]/b_{11} \\ f_{22} &= [\partial_x^3 f_{12} + (1 - b_{11})\partial_x f_{12}]/b_{22} \end{aligned} \tag{4.20}$$

where a_1, a_2, a_3, and a_4 are (possibly complex) constants. The constraints on the constants required to eliminate spurious solutions turn out to be

$$a_4^2 - a_3^2 = a_2^2 - a_1^2 = 1/d\,. \tag{4.21}$$

This construction provides solutions to (4.10) when the four roots mentioned above are distinct, and when all of the various denominators are non-vanishing. This group of solutions, which includes 'almost all' solutions, can be succinctly characterized as those for which the constants satisfy the inequality

$$d\,b_{11}b_{22}(b_{11}+b_{22}-1)\neq 0 \tag{4.22}$$

and will be referred to as the *generic* solution for our special case of (4.10). Non-generic solutions can be constructed in a similar way. It is easy to see that for any choice of B such that (4.22) is satisfied, there is a two-parameter family of choices for a_1, a_2, a_3, and a_4 for which F, and thus J_1, are real, so the generic solution contains four constants constrained only by the inequality (4.22).

For concreteness we will give a particularly simple specific example. If we choose $m=1$, $n=\mathrm{i}$, $b_{11}=(1+\sqrt{5})/2$, $b_{22}=(1-\sqrt{5})/2$, $a_1=a_3=0$ and $a_2=a_4=1/2$, then the solution to (4.11) is

$$\begin{aligned}(J_1)_{11}&=-\tfrac14\big[(1+\sqrt{5})\cos x+(\sqrt{5}-3)\cosh x-4\big]/\gamma\\(J_1)_{12}&=(\sin x+\sinh x)/\gamma\\(J_1)_{22}&=\tfrac14\big[(\sqrt{5}-1)\cos x+(\sqrt{5}+3)\cosh x-4\big]/\gamma\end{aligned} \tag{4.23}$$

where

$$\gamma=-\sin x\sinh x-2\cos x\cosh x+\sqrt{5}(\cos x+\cosh x)-2\,. \tag{4.24}$$

To complete our calculations we also want to give solutions to (4.9). It is easy to show by direct substitution that if F and J_1 satisfy (4.12) and (4.10), then F^{-1} satisfies (4.9), and so the above construction also provides us with a generic solution of the two-element example. For the particular case in which the three-element J_1 is given by (4.23) the corresponding two-element solution to (4.9) is the J_1 whose elements are

$$\begin{aligned}(J_1)_{11}&=\tfrac14(1+\sqrt{5})\big[(1+\sqrt{5})\sin x+(3-\sqrt{5})\sinh x\big]/\delta\\(J_1)_{12}&=n(\cos x+\cosh x)/\delta\\(J_1)_{22}&=\tfrac14(\sqrt{5}-1)\big[(1-\sqrt{5})\sin x+(3+\sqrt{5})\sinh x\big]/\delta\end{aligned} \tag{4.25}$$

where

$$\delta=-2\sin x\sinh x+\cos x\cosh x+1\,. \tag{4.26}$$

The progressing-wave solutions to the systems of wave equations in these two examples are given by (4.1), specialized to the case $k_1=1, k_0=-1$ with J_1 given by (4.23) and $J_0=I$, and specialized to the case $k_1=1, k_0=0$ with $J_1=J_0^{-1}$ given by (4.25), respectively.

It is worth noting that Bäcklund transformations, as well as some particular 2×2 solutions, were given by Andreev [28, 29] for a class of finite free-ended non-Abelian lattice motions defined by the pair of (equivalent) *subsidiary first-order differential* conditions

$$\sum_k(\partial_u J_k)J_k^{-1}=0 \qquad \sum_k J_{-1}(\partial_v J_k)=0 \tag{4.27}$$

where the sum extends over all the elements in the lattice. His results are of general interest in the context of this paper, but if his subsidiary conditions are imposed simultaneously with our assumption of a symmetric J_1, it follows that in the specialized case which we are considering, C in (4.11) vanishes, and the resulting motion can be gauged to one with all the J_k diagonal. Thus if his results are specialized to correspond to self-adjoint systems of wave equations those equations are essentially uncoupled, and so his results complement, rather than overlap with, ours.

5. Conclusion and open problems

The formalism developed in this paper raises a number of interesting questions. The first one, which we shall discuss in some detail, is of a formal nature as it concerns a possible refinement of the definitions of self-adjointness and almost self-adjointness for systems of linear wave equations. We saw in section 2 that, in the Abelian case, if a Toda-lattice motion of any length has a fixed centre, i.e. if $j_0 = 1$, then the motion is anti-symmetrical, i.e. $j_k = j_{-k}^{-1}$, $k \in \mathrm{Z}$, and at the same time the corresponding wave equation $\partial_{uv}^2\phi - j_1\phi = 0$ is formally self-adjoint for arbitrary j_1. It is natural to look for a non-Abelian generalization of this pleasing correspondence, but the situation is more complicated in the matrix case. In the example considered in section 4 it was first assumed that $J_0 = I$, implying that $J_1 = J_{-1}^{-1}$, a first step in characterizing a non-Abelian motion to be 'anti-symmetrical', and independently assumed that $J_1 = J_1^{\mathrm{T}}$, with the consequence that the coupled system of wave equations, with the two coefficient matrices I and J_1, was formally self-adjoint. Regardless of the symmetry properties of J_1, the demand that $J_2 = J_{-2}^{-1}$, i.e. the stipulation that the motion be 'anti-symmetrical' at least as far as the second elements on either side of the fixed centre, resulted in the first-order differential condition (4.4) on J_1, which is not identically satisfied in the non-Abelian case. One can in fact show that given

$$J_0 = I \qquad J_1 = J_{-1}^{-1} \tag{5.1}$$

the condition

$$J_k = J_{-k}^{-1} \qquad 2 \leqslant k \leqslant K \tag{5.2}$$

with integer $K \geqslant 2$, is equivalent to

$$(\partial_v J_{k-1})J_{k-1}^{-1}(\partial_u J_{k-1}) = (\partial_u J_{k-1})J_{k-1}^{-1}(\partial_v J_{k-1}) \qquad 2 \leqslant k \leqslant K \tag{5.3}$$

modulo the lattice dynamics. Thus a non-Abelian motion can be 'anti-symmetrical' out to K elements on either side of the centre, where K is a positive integer, but not beyond, and furthermore this is equivalent to $K-1$ differential conditions on J_1. It can also be shown that if instead of (5.1) we assume

$$J_0 = J_0^{\mathrm{T}} \qquad J_1 = J_1^{\mathrm{T}} \tag{5.4}$$

then (5.3) are equivalent to

$$J_k = J_k^{\mathrm{T}} \qquad 2 \leqslant k \leqslant K\,. \tag{5.5}$$

Thus *either* of the assumptions (5.1) and (5.4) is propagated to $\pm K$ elements along the lattice, in the sense of (5.2) and (5.5) respectively, if and only if the *same* set of $K-1$ conditions, (5.3), are satisfied by J_1. This suggests that although either (5.1) and (5.2) or (5.4) and (5.5) may single out an interesting subset of lattice motions, the simultaneous imposition of both $J_k = J_{-k}^{-1}$ *and* $J_k = J_k^{\mathrm{T}}$, $0 \leqslant k \leqslant K$, produces a more fundamental subset of motions. Support for this notion follows from a consideration of the corresponding systems of linear wave equations.

Since by definition the system of equations (3.11) is formally self-adjoint when it is of the form

$$(\partial_v D \partial_u - J_1)\Phi_1 = 0 \qquad (5.6)$$

with D a constant symmetric matrix and J_1 any symmetric matrix, we see that a non-Abelian Toda-lattice motion that satisfies both (5.1) and (5.2), and (5.4) and (5.5), corresponds to a proper subset of the formally self-adjoint systems of equations that we might call the *differentiably self-adjoint systems of degree* K, i.e. those for which $J_0 = D = I$, and J_1 is not only symmetric but satisfies the $K-1$ differential conditions included in (5.3). It is only by means of this refinement of self-adjointness that one can maintain the correspondence between anti-symmetry of Toda-lattice motions and self-adjointness of wave equations in the non-Abelian case. Note that a constant and symmetric D in (5.6) can be gauged to I while preserving the symmetry of J_1 if and only if it commutes with J_1. Thus the algebraic condition in the refinement, $D = I$, has content, although the set of differential conditions is more striking. There is an analogous notion of *differentiably almost self-adjoint of degree* K for systems of coupled wave equations based on 'anti-symmetrical' non-Abelian motions without a centre that involves the same conditions (5.3), but we will not discuss the details here. Whether or not any of these generalizations or refinements of formal self-adjointness is actually of significance for the theory of linear differential operators is an open question.

There are also interesting unsolved problems concerning the free-ended non-Abelian finite-lattice motions. Despite the simplicity of the general solution of the scalar Liouville equation, the matrix generalization of that ubiquitous equation seems not so easily dealt with. In section 4 we gave its generic solution in the case when the dependent variable is a symmetric 2×2 matrix depending on u and v, only in the combination $u + v$, and it was a non-trivial construction. This is an extremely special example of a finite-lattice motion, and it would be of interest to construct wider classes of solutions to the matrix Liouville equation and, more generally, to find a variety of motions of longer finite lattices.

The corresponding exactly-solvable truly-coupled systems of wave equations thereby obtained could be of interest in their own right, with applications of their progressing-wave solutions to various fields on curved spacetimes being a case in point. If, for example, one examines perturbations of the multicomponent gravitational field for a fixed background spacetime, one naturally obtains a coupled system of linear wave equations. The results reviewed in the introduction concerning such perturbations were restricted to background spacetimes that are both homogeneous and isotropic, i.e. to cosmological spacetimes, because for this class it was possible to find a single field, governed by a single equation, that could serve as a potential for the full set of components. Thus exact solvability of a single equation gave exact results for the system governing the full set of perturbations. Given significant families of exactly solvable systems of coupled equations it may be possible to find spacetimes possessing much less symmetry whose gravitational perturbations are governed by sets of wave equations that cannot be decoupled, but that are exactly solvable in terms of progressing waves, thereby greatly extending these results.

We will conclude by discussing a possible application with classical roots, but with implications for problems of much current interest in mathematical physics. In a well known 1949 paper Bargmann gave a set of simple potentials for the Schrödinger equation for which the reflection coefficient vanished [34], and a few years later Kay and Moses gave the set of all non-singular reflectionless potentials for the one dimensional Schrödinger equation [35]. Reflectionless potentials for the Schrödinger equation are easily related to wave equations with progressing-wave solutions [36], and this will presumably generalize to a connection between systems of exactly solvable wave equations and sets of reflectionless potentials for multichannel scattering processes. There is some literature on the effort to construct such sets of potentials [37–39]. Given the connection provided by the inverse scattering method between reflectionless potentials for single Schrödinger equations and multisoliton solutions of equations such as the KdV equation, the matter would seem to be of increasing interest at the present time, since as methods for the construction of motions of the finite free-ended non-Abelian Toda lattice evolve, corresponding sets of reflectionless potentials may lead to the construction of multisoliton solutions for matrix-valued nonlinear evolution equations.

Acknowledgment

This work has been supported in part by the Natural Sciences and Engineering Research Council of Canada.

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